

An Eulerian permutation statistic and generalizations

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Abstract

Recently, the second author studied an Eulerian statistic (called *cover*) in the context of convex polytopes, and proved an equal joint distribution of $(\text{cover}, \text{des})$ with (des, exc) . In this paper, we present several direct bijective proofs that *cover* is Eulerian, and examine its generalizations and their Mahonian partners. We also present a quasi-symmetric function proof (suggested by Michelle Wachs) of the above equal joint distribution.

1 Introduction

Permutation statistics is well explored subject in mathematics. MacMahon [8] considered four different statistics for a permutation: the number of *descents* (des), the number of *exceedances* (exc), the number of *inversions* (inv), and the *major index* (maj). For a permutation $\sigma \in \mathfrak{S}_n$:

$$\begin{aligned}\text{DES}(\sigma) &= \{i : \sigma(i) > \sigma(i-1)\} \\ \text{des}(\sigma) &= |\text{DES}(\sigma)| \\ \text{maj}(\sigma) &= \sum_{i \in \text{DES}(\sigma)} i\end{aligned}$$

Meanwhile, exc and inv are defined:

$$\begin{aligned}\text{exc}(\sigma) &= \#\{i : \sigma(i) > i\} \\ \text{inv}(\sigma) &= \#\{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}\end{aligned}$$

It is first due to MacMahon, using an algebraic proof, that exc is equidistributed with des , and that inv is equidistributed with maj , over \mathfrak{S}_n , i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{exc}(\sigma)}, \quad \sum_{\sigma \in \mathfrak{S}_n} x^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma)}$$

Any permutation statistic that is equidistributed with des is said to be *Eulerian* and a permutation statistic that is equidistributed with inv is said to be *Mahonian* (see [2]). We call a pair of statistics *Euler-Mahonian* if one is Eulerian and one Mahonian [2]. In the last thirty years, people have studied many new statistics, especially those that are Eulerian and Mahonian, for example [1, 3, 4, 5, 9]. There are also many connections with other areas, for example [11, 15].

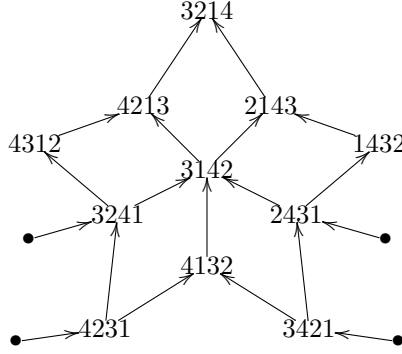
1.1 Cover is Eulerian

The second author recently defined a new Eulerian permutation statistic called “*cover*” motivated by her study of the h -polynomial of the half-open hypersimplices [7].

There she defined a poset on \mathfrak{S}_n representing the triangulation of a hypersimplex, and $\text{cover}(\sigma)$ as the number of elements σ covers in the poset. It is then shown that cover can be computed as

$$\text{cover}(\sigma) = \#\{i \in [n-1] \mid \sigma^{-1}(i) < \sigma^{-1}(i+1)\} + \begin{cases} 0 & \text{if } \sigma(1) = 1 \\ 1 & \text{if } \sigma(1) \neq 1 \end{cases} \quad (1.1)$$

For example, part of the poset for $n = 4$ is shown below, where each arrow represents a cover relation in the poset and points toward the larger permutation; the bullets represent other permutations not displayed.



We can see that the cover of $\sigma = 3\ 1\ 4\ 2$ is 3, either by looking at the poset or using definition (1.1), and seeing that we have $\sigma^{-1}(2) > \sigma^{-1}(1) + 1$, $\sigma^{-1}(4) > \sigma^{-1}(3) + 1$, and $\sigma(1) = 3 \neq 1$. Or, if we take $\sigma = 2\ 4\ 3\ 1$, we see that $\text{cover}(\sigma) = 2$, because $\sigma^{-1}(3) > \sigma^{-1}(2) + 1$, and $\sigma(1) = 2 \neq 1$.

We do not go into details about the poset or the hypersimplices here; the definition (1.1) will be enough for this paper. However; using two different shellable triangulations of the half-open hypersimplices, the following identity can be proved.

Proposition 1.1. [7]

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} x^{\text{cover}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} x^{\text{des}(\sigma)},$$

From the above result, we can see that cover is Eulerian. We will provide several more direct proofs that cover is Eulerian in the paper. In Section 2 we will compare $\widetilde{\text{des}}_2$ with exc , and in Section 3, we will use bijections motivated by the Mahonian statistics to show that $\widetilde{\text{des}}_2$ is equidistributed with des .

1.2 Mahonian partner

Consider the following variants on the statistic des . The simplest is the number of *ascents*, denoted asc :

$$\begin{aligned} \text{ASC}(\sigma) &= \{i : \sigma(i) < \sigma(i+1)\} \\ \text{asc}(\sigma) &= |\text{ASC}(\sigma)| \end{aligned}$$

We also consider *2-descents*:

$$\begin{aligned} \text{DES}_2(\sigma) &= \{i : \sigma(i) \geq \sigma(i+1) + 2\} \\ \text{des}_2(\sigma) &= |\text{DES}_2(\sigma)| \end{aligned}$$

We also define a major version,

$$\text{maj}_2(\sigma) = \sum_{i \in \text{DES}_2(\sigma)} i$$

Finally, we define the following two variants on des_2 and maj_2 :

$$\widetilde{\text{des}}_2(\sigma) = \begin{cases} \text{des}_2(\sigma) & \text{if } \sigma(1) = n \\ \text{des}_2(\sigma) + 1 & \text{if } \sigma(1) \neq n \end{cases} \quad (1.2)$$

$$\widetilde{\text{maj}}_2(\sigma) = \text{maj}_2(\sigma) + \text{asc}(\sigma^{-1}) \quad (1.3)$$

It turns out that $\widetilde{\text{des}}_2$ has a very natural relationship with cover . For any permutation $\sigma \in \mathfrak{S}_n$, define $\sigma' \in \mathfrak{S}_n$ by $\sigma'(i) = n + 1 - \sigma^{-1}(i)$. Then it is easy to see that $\text{cover}(\sigma) = \widetilde{\text{des}}_2(\sigma)$. Thus, for the rest of the

paper we will be considering $\widetilde{\text{des}}_2$ rather than cover , due to its close analog with des . In Section 3, we will show that

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} y^{\text{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\widetilde{\text{des}}_2(\sigma)} y^{\widetilde{\text{maj}}_2(\sigma)} \quad (1.4)$$

From the above result, we can see that $\widetilde{\text{maj}}_2$ is a Mahonian partner for $\widetilde{\text{des}}_2$. In this paper, we will provide a bijective proof for that in Section 3. The bijection is mainly the one used in [10] which shows that the statistic we will call $\widetilde{\text{maj}}_k$ is Mahonian, but we will also use techniques from [14]. To define $\widetilde{\text{maj}}_k$, we first define k -analogs of des : The statistic $\widetilde{\text{maj}}_k$ is defined as follows:

$$\text{DES}_k(\sigma) = \{i : \sigma(i) \geq \sigma(i+1) + k\} \quad (1.5)$$

$$\text{des}_k(\sigma) = |\text{DES}_k| \quad (1.6)$$

$$\widetilde{\text{des}}_k(\sigma) = \begin{cases} \text{des}_k(\sigma) & \text{if } \sigma(1) > n + 1 - k \\ \text{des}_k(\sigma) + 1 & \text{otherwise} \end{cases} \quad (1.7)$$

and then we define

$$\text{maj}_k(\sigma) = \sum_{i \in \text{DES}_k(\sigma)} i \quad (1.8)$$

$$\widetilde{\text{maj}}_k(\sigma) = \text{maj}_k(\sigma) + \{(i, j) : i < j \text{ and } \sigma(i) < \sigma(j) < \sigma(i) + k\} \quad (1.9)$$

It should be noted that these definitions are consistent with those of des , des_2 , $\widetilde{\text{des}}_2$, maj_2 , and $\widetilde{\text{maj}}_2$.

We will also prove a generalization of (1.4), namely that

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}_k(\sigma)} y^{\widetilde{\text{maj}}_k(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\widetilde{\text{des}}_{k+1}(\sigma)} y^{\widetilde{\text{maj}}_{k+1}(\sigma)}.$$

2 A bijection for exc and $\widetilde{\text{des}}_2$

2.1 A map taking exc to des

In order to show that $\widetilde{\text{des}}_2$ is Eulerian, we will present a bijection which takes the statistic exc to $\widetilde{\text{des}}_2$. In order to understand the motivation for this bijection, we first present a bijection which takes exc to des , given in [16], which we will call cycle_0 . It is also known as the Foata map [3].

For a permutation $\sigma \in \mathfrak{S}_n$, define the *standard cycle notation* of σ to be

$$\sigma = (a_{1,1} \ a_{1,2} \ \cdots \ a_{1,\ell(1)})(a_{2,1} \ a_{2,2} \ \cdots \ a_{2,\ell(2)}) \ \cdots \ (a_{k,1} \ a_{k,2} \ \cdots \ a_{k,\ell(k)})$$

where (i) $\sigma(a_{i,j}) = a_{i,j+1}$ for $1 \leq i \leq k$, and $1 \leq j \leq \ell(i) - 1$, (ii) $\sigma(a_{i,\ell(i)}) = a_{i,1}$, (iii) each number in $\{1, \dots, n\}$ is equal to $a_{i,j}$ for exactly one pair (i, j) , and (iv) $a_{1,1} < a_{2,1} < \cdots < a_{k,1}$, but $a_{i,1} > a_{i,j}$ for all i and $j \geq 2$. For example, the standard cycle notation of the permutation $\sigma = 4 \ 5 \ 6 \ 1 \ 2 \ 7 \ 8 \ 3$ is $(41)(52)(8367)$.

Now, given a permutation σ , write the standard cycle notation of σ^{-1} , and erase the parentheses. This gives a permutation π . Thus, if we take $\sigma = 3 \ 4 \ 1 \ 5 \ 2$, the standard cycle notation of σ^{-1} is $(3 \ 1)(5 \ 4 \ 2)$, so $\pi = 3 \ 1 \ 5 \ 4 \ 2$.

We can reverse this map as well. Given π , there is a unique way to split it into cycles such that it satisfies condition (iv) of standard cycle notation. We simply find each element of π that is larger than each element before it. Such elements are known as the *left-to-right* maxima of π [16]. We let the left-to-right maxima be the first elements of the cycles. For instance, if $\pi = 3 \ 4 \ 2 \ 1 \ 5$, then the left-to-right maxima are 3, 4, and 5, so we partition it into $(3)(4 \ 2 \ 1)(5)$, and then we get $\sigma = 2 \ 4 \ 3 \ 1 \ 5$.

It is not hard to see that for any $\sigma \in \mathfrak{S}_n$, we have $\text{exc}(\sigma) = \text{des}(\pi)$, as in the example.

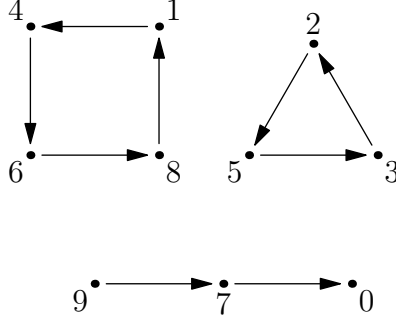


Figure 1: The graph G for $\sigma = 7\ 8\ 3\ 5\ 1\ 2\ 4\ 9\ 6$ in the $cycle_{-1}$ bijection

2.2 A map between exc and $\widetilde{\text{des}}_2$

We want to create a bijection between exc and $\widetilde{\text{des}}_2$. It will be slightly more convenient to construct the bijection exc to a slightly different statistic. Define

$$\widetilde{\text{des}}'_k(\sigma) = \begin{cases} \text{des}_k(\sigma) & \text{if } \sigma(n) < k \\ \text{des}_k(\sigma) + 1 & \text{if } \sigma(n) \geq k \end{cases}$$

Let $flip$ be the bijection which takes σ to σ' , defined by $\sigma'(i) = n + 1 - \sigma(n + 1 - i)$. By comparing the definition of $\widetilde{\text{des}}'_k$ with the definition of $\widetilde{\text{des}}_k$ (see (1.2)), we see that the involution $flip$ takes one statistic to the other (i.e., $\widetilde{\text{des}}_k(\sigma) = \widetilde{\text{des}}'_k(\sigma')$). We will construct a bijection based on the Foata map which takes exc to $\widetilde{\text{des}}'_2$. By composing this bijection with $flip$, we obtain a bijection that takes exc to $\widetilde{\text{des}}_2$. Note that while we could have chosen this as the definition of $\widetilde{\text{des}}_k$ to begin with, it is our original definition that will be more natural in Section 3.

Now we describe our bijection, which we will call $cycle_{-1}$. Take any permutation $\sigma \in \mathfrak{S}_n$. Construct a graph G as follows: let its vertices be the set $\{0, \dots, n\}$. For each $i \in \{1, \dots, n\}$, draw a directed edge from $\sigma(i)$ to $i - 1$. For instance, the graph G for $\sigma = 7\ 8\ 3\ 5\ 1\ 2\ 4\ 9\ 6$ is given in Figure 1.

Each vertex of G except 0 will have exactly one outgoing edge, and each vertex except n will have exactly one incoming edge. Thus, G consists of some cycles and a path from n to 0. Define the *standard cycle-path notation* of G as follows: first write the standard cycle notation (as defined in Section 2.1) of the cycles, and then write the path from n to 0. For example, the standard cycle-path notation of G in our example is $(5\ 3\ 2)(8\ 1\ 4\ 6)(9\ 7\ 0)$. Finally, we let π be the permutation obtained by removing the parentheses and removing the trailing 0. Hence, in this case, $\pi = 5\ 3\ 2\ 8\ 1\ 4\ 6\ 9\ 7$. This defines our mapping $cycle_{-1}$. Note that we can reverse this map in the same way we reversed $cycle_0$, the foata map; that is, we just partition π into the cycles, and one path, by looking at the left-to-right maxima and making these the starts to the cycles and the path. (In particular n , will be the first element of the path).

We want to show that if $cycle_{-1}$ takes σ to π , then $\text{exc}(\sigma) = \widetilde{\text{des}}'_2(\pi)$, thus proving that $\widetilde{\text{des}}_2$ is Eulerian. In fact, we will prove a stronger statement. Define

$$\text{exc}_k(\sigma) = \{i : \sigma(i) \geq i + k\}$$

Then,

Proposition 2.1. *For $k \geq 1$, if $cycle_{-1}$ maps σ to π , then $\text{exc}_k(\sigma) = \widetilde{\text{des}}'_{k+1}(\pi)$.*

For $k = 1$, we get that $cycle_{-1}$ maps exc to $\widetilde{\text{des}}'_2$.

Proof of Proposition 2.1. Suppose we extend π to $\hat{\pi}$ such that it has a 0 at the end (i.e., $\hat{\pi}(n+1) = 0$), as if we never removed 0 from the end of the standard cycle-path notation. Then, the case where $\hat{\pi}(n) \geq k+1$ becomes a $(k+1)$ -descent. We will show that we can find a mapping between the k -exceedances of σ and the $(k+1)$ -descents of this extended $\hat{\pi}$.

Suppose we have a k -exceedance $\sigma(i) \geq i+k$. In G , $\sigma(i)$ points to $i-1$, which is at most $\sigma(i) - (k+1) \leq \sigma(i)$. Now, $i-1$ is not the largest element in its cycle or path, since $\sigma(i)$ is in the same path, so in the standard cycle-path notation, $\sigma(i)$ comes directly before $i-1$, giving a $(k+1)$ -descent.

Conversely, suppose we have a $(k+1)$ -descent $\hat{\pi}(i) \geq \hat{\pi}(i+1) + k+1$. Now, it cannot be the case that $\hat{\pi}(i+1)$ is at the start of a cycle or path when $\hat{\pi}$ is decomposed into cycles and a path, since $\hat{\pi}(i)$ is larger than $\hat{\pi}(i+1)$. Therefore, $\hat{\pi}(i)$ and $\hat{\pi}(i+1)$ are in the same cycle or path, and so $\sigma(\hat{\pi}(i+1) + 1) = \hat{\pi}(i)$. (Recall that G is such that $\sigma(j)$ points to $j-1$.) This is a k -exceedance. \square

We will now show another interesting property of this map. Define the statistic unexc by

$$\text{unexc}(\sigma) = \#\{i : \sigma(i) < i\}$$

Then,

Proposition 2.2. *If cycle_{-1} maps σ to π , then $\text{unexc}(\sigma) = \text{asc}(\pi)$.*

Proof. We will map unexceedances of σ to the ascents of π .

Suppose $\sigma(i) < i$ is an unexceedance. If $\sigma(i) = i-1$, then $\sigma(i)$ points to $\sigma(i)$ in G . In standard cycle-path notation, $\sigma(i)$ will be in a cycle by itself, and the first element of the next cycle or path will be larger than it, so this is an ascent. If $\sigma(i) < i-1$, then we have $\sigma(i)$ pointing to $i-1$, which is larger than $\sigma(i)$. Now, $\sigma(i)$ and $i-1$ will be in the same cycle or path. Either $\sigma(i)$ comes directly before $i-1$, in which case we have an ascent immediately; otherwise, $\sigma(i)$ is at the end of some cycle and $i-1$ is at the beginning. But we always have an ascent from one cycle to the next, so even in this case, $\sigma(i)$ is still part of an ascent.

Now we do the reverse direction; suppose $\pi(i) < \pi(i+1)$. We have two cases. If $\pi(i)$ and $\pi(i+1)$ are in the same cycle or path when π is partitioned, then we simply have $\sigma(\pi(i+1) + 1) = \pi(i) < \pi(i+1) + 1$, which is an unexceedance. Otherwise, $\pi(i)$ is at the end of its cycle, and $\pi(i+1)$ is at the beginning of the next cycle or path. In G , we know $\pi(i)$ will point to the element at the beginning of its own cycle, which is at least as large as $\pi(i)$ (possibly the same, if the cycle is of length one). Call the element at the beginning of this cycle $\pi(j)$. Then we will have $\sigma(\pi(j) + 1) = \pi(i) < \pi(j) + 1$, which is again an unexceedance. \square

Now, not only do we know that $\widetilde{\text{des}}_2(\sigma)$ is Eulerian, but the following corollaries are immediate from the Propositions 2.1 and 2.2:

Corollary 2.3. *The multivariate statistic $(\text{unexc}, \text{exc}, \text{exc}_2, \text{exc}_3 \dots)$ is equidistributed with the multivariate statistic $(\text{asc}, \widetilde{\text{des}}_2, \widetilde{\text{des}}_3, \widetilde{\text{des}}_4 \dots)$.*

Proof. The first statistic is taken to the second by applying the mapping cycle_{-1} followed by the mapping of flip . For instance, continuing with the example of $\sigma = 7 \ 8 \ 3 \ 5 \ 1 \ 2 \ 4 \ 9 \ 6$, we apply cycle_{-1} to get $\pi = 5 \ 3 \ 2 \ 8 \ 1 \ 4 \ 6 \ 9 \ 7$ (as stated previously), and then applying flip we get $\pi' = 3 \ 1 \ 4 \ 6 \ 9 \ 2 \ 8 \ 7 \ 5$. \square

Corollary 2.4. *$(\text{asc}, \widetilde{\text{des}}_2)$ has the same distribution as $(\widetilde{\text{des}}_2, \text{asc})$*

Proof. $(\text{asc}, \widetilde{\text{des}}_2)$ has the same distribution as $(\text{unexc}, \text{exc})$. By the involution flip , we see that $(\text{unexc}, \text{exc})$ has the same distribution as $(\text{exc}, \text{unexc})$. The result follows. \square

3 Mahonian statistics

3.1 Codes

codes-sec

In order to study Mahonian statistics such as maj , we will use the notion of a *code*. A code c of length n is a sequence of integers $c(1), \dots, c(n)$ such that $0 \leq c(i) < i$ for all i . We let \mathfrak{C}_n denote the set of codes of length n ; it is clear that there are $n!$ such codes.

Codes, as examined by Skandera [14], provide a natural way to think about the Mahonian distribution. Define the statistic \sum by

$$\sum(c) = c(1) + \dots + c(n) \quad (3.1)$$

for $c \in \mathfrak{C}_n$. Then the statistic \sum is Mahonian. In [14], it is shown that sum over \mathfrak{C}_n has the same distribution as maj and inv over \mathfrak{S}_n .

Before looking at maj in the next section, let us consider the simpler case of inv , as shown in [14], to understand how we can use codes to study Mahonian statistics. Consider any permutation $\sigma \in \mathfrak{S}_n$. We construct a code $c \in \mathfrak{C}_n$ as follows. Simply let

$$c(i) = \#\{j : j > \sigma^{-1}(i), \sigma(j) < i\}. \quad (3.2)$$

Then $\text{inv}(\sigma) = \sum(c)$. We call this mapping an inv -coding scheme, and we denote it by **Inv-Code**. It is not difficult to see that this map is a bijection; indeed, we can reverse it in this way: to construct σ from c , start with the permutation 1. Then, for $i \geq 2$, insert i into the permutation at the $c(i)^{\text{th}}$ spot from the end (the 0^{th} position being the end, the $(i-1)^{\text{th}}$ position being the beginning). For instance, if we have 4 1 2 3 and $c(5) = 3$, we would insert 5 to get 4 5 1 2 3.

Skandera also considers another statistic on codes, st . He defines st to be the length of the longest sequence $p_1 < p_2 < \dots < p_\ell$ such that $c(p_i) > i$ for all i . For our purposes it will be more convenient to use the following inductive definition, which is easily seen to be equivalent. Let $\text{st}(0) = 0$, and for $n \geq 2$ and $c \in \mathfrak{C}_n$, let

$$\text{st}(c) = \begin{cases} \text{st}(c_{n-1}) & \text{if } c(n) \leq \text{st}(c_{n-1}) \\ \text{st}(c_{n-1}) + 1 & \text{if } c(n) > \text{st}(c_{n-1}) \end{cases} \quad (3.3)$$

where $c_{n-1} \in \mathfrak{C}_{n-1}$ is the prefix $c(1), \dots, c(n-1)$ of c .

Skandera then shows that the distribution of st over \mathfrak{C}_n is Eulerian. He does this by constructing a maj -coding scheme, that is, a bijection $\widetilde{\text{maj-Code}} : \mathfrak{S}_n \rightarrow \mathfrak{C}_n$ for which $\text{maj}(\sigma) = \sum(\widetilde{\text{maj-Code}}(\sigma))$. The bijection also has the property that $\text{des}(\sigma) = \text{st}(\widetilde{\text{maj-Code}}(\sigma))$, and thus has the same distribution over \mathfrak{C}_n as des does over \mathfrak{S}_n . We will provide such a bijection in this paper; however, it will be a slightly modified version of Skandera's bijection.

Rawlings [10] generalizes the bijection. He provides, for any constant k , what we will call a $\widetilde{\text{maj}_k}$ -coding scheme, a bijection $\widetilde{\text{maj}_k\text{-Code}} : \mathfrak{S}_n \rightarrow \mathfrak{C}_n$, such that

$$\widetilde{\text{maj}_k}(\sigma) = \sum(\widetilde{\text{maj}_k\text{-Code}}(\sigma)). \quad (3.4)$$

This shows that maj_k is Mahonian for any k .

3.2 Our results

We take these ideas a step further, by combining Rawlings' bijections with analysis of the st statistic. First, we generalize the st statistic to a more general st_k (such that st_1 is identical to st). Again, for any k , we define st_k inductively. Let $\text{st}_k(0) = 0$, and for $n \geq 2$ and $c \in \mathfrak{C}_n$, we let

$$\text{st}_k(c) = \begin{cases} \text{st}_k(c) & \text{if } c(n) \leq \text{st}_k(c_{n-1}) + k - 1 \\ \text{st}_k(c_{n-1}) + 1 & \text{if } c(n) > \text{st}_k(c_{n-1}) + k - 1 \end{cases} \quad (3.5)$$

where, again $c_{n-1} \in \mathfrak{C}_{n-1}$ is the prefix $c(1), \dots, c(n-1)$ of c .

We will present a specific set of bijections $\widetilde{\text{maj}}_k\text{-Code} : \mathfrak{S}_n \rightarrow \mathfrak{C}_n$ using essentially the same technique as in [10]. In particular, we will show, by construction,

Theorem 3.1. *There exists a set of bijections $\widetilde{\text{maj}}_k\text{-Code} : \mathfrak{S}_n \rightarrow \mathfrak{C}_n$, such that, for all k and every $\sigma \in \mathfrak{S}_n$*

- (i) $\widetilde{\text{maj}}_k(\sigma) = \sum(\widetilde{\text{maj}}_k\text{-Code}(\sigma))$
- (ii) $\text{des}_k(\sigma) = \text{st}_k(\widetilde{\text{maj}}_k\text{-Code}(\sigma))$
- (iii) $\widetilde{\text{des}}_{k+1}(\sigma) = \text{st}_k(\widetilde{\text{maj}}_{k+1}\text{-Code}(\sigma))$

This immediately gives the following result:

Corollary 3.2. *For any k , the bivariate statistics $(\text{des}_k, \widetilde{\text{maj}}_k)$ and $(\widetilde{\text{des}}_{k+1}, \widetilde{\text{maj}}_{k+1})$ have the same distribution.*

Proof. By considering the bijection $\widetilde{\text{maj}}_k\text{-Code}$, we see that $(\text{des}_k, \widetilde{\text{maj}}_k)$ has the same distribution as (st_k, \sum) . By considering the bijection $\widetilde{\text{maj}}_{k+1}\text{-Code}$, we see that $(\widetilde{\text{des}}_{k+1}, \widetilde{\text{maj}}_{k+1})$ has the same distribution as $(\text{st}_k, \text{sum})$. Thus, $(\text{des}_k, \widetilde{\text{maj}}_k)$ and $(\widetilde{\text{des}}_{k+1}, \widetilde{\text{maj}}_{k+1})$ have the same distribution. \square

For the case $k = 1$, Corollary 3.2 gives equation (1.4) and thus gives another proof that $\widetilde{\text{des}}_2$ is Eulerian.

3.3 The $\widetilde{\text{maj}}_k$ -coding scheme

We will first describe what the mapping $\widetilde{\text{maj}}_k\text{-Code}$ is, and then prove things about it. The map itself is very easy to state, although it is not initially obvious that it is a bijection, or even that its output will always lie in \mathfrak{C}_n .

Define the operation $\text{cut} : \mathfrak{S}_n \rightarrow \mathfrak{S}_{n-1}$ as follows. For any $\sigma \in \mathfrak{S}_n$, $\text{cut}(\sigma)$ is the permutation obtained by removing 1 from σ and decrementing all other numbers. For instance, if $\sigma = 4\ 1\ 5\ 2\ 3$, we remove 1 to get $4\ 5\ 2\ 3$ and $\text{cut}(\sigma) = 3\ 4\ 1\ 2$.

Mapping 3.3 ($\widetilde{\text{maj}}_k\text{-Code}$). *Take any permutation $\sigma \in \mathfrak{S}_n$. Let $\sigma_n = \sigma$ and $\sigma_i = \text{cut}(\sigma_{i+1})$ for $1 \leq i \leq n-1$. Thus we have a sequence $\sigma_n, \dots, \sigma(1)$ with $\sigma_i \in \mathfrak{S}_i$ for all i . Now let $c(1) = 0$ and $c(i) = \widetilde{\text{maj}}_k(\sigma_i) - \widetilde{\text{maj}}_k(\sigma_{i-1})$ for $i \geq 2$. Then we let $\widetilde{\text{maj}}_k\text{-Code}$ be the map which takes σ to c .*

Now it is immediate that $\widetilde{\text{maj}}_k(\sigma) = \sum(\widetilde{\text{maj}}_k\text{-Code}(\sigma))$. Thus, if we can prove that this mapping is a bijection $\mathfrak{S}_n \rightarrow \mathfrak{C}_n$, then we will already satisfy (i) of Theorem 3.1.

To understand this map, we examine what happens when we go from $\text{cut}(\sigma)$ to σ . Let us define an inverse of the cut operation as follows. For any permutation σ , let $\text{insert}_i(\sigma)$ be the permutation obtained by incrementing each number of σ by 1 and then inserting 1 at position i . Thus, $\text{insert}_i(\sigma)(i) = 1$ and $\text{cut}(\text{insert}_i(\sigma)) = \sigma$ for any i .

We want to study $\widetilde{\text{maj}}_k(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_k(\sigma)$, for $\sigma \in \mathfrak{S}_{n-1}$. We claim that as i ranges over $\{1, \dots, n\}$, this quantity ranges over $\{0, \dots, n-1\}$. Once we prove this, it will be clear that $\widetilde{\text{maj}}_k$ is a surjection \mathfrak{C}_n . Indeed, consider any code $c \in \mathfrak{C}_n$. Starting with $\sigma_1 = 1$, we can find a sequence $\sigma_1, \dots, \sigma_n$ such that $\text{cut}(\sigma_i) = \sigma_{i-1}$ and $\widetilde{\text{maj}}_k(\sigma_i) - \widetilde{\text{maj}}_k(\sigma_{i-1}) = c(i)$. Then $\widetilde{\text{maj}}_k\text{-Code}(\sigma_n) = c$.

Our analysis of the value of $\widetilde{\text{maj}}_k(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_k(\sigma)$ relies on a set which we will call $A_k(\sigma)$.

$$A_k(\sigma) \stackrel{\text{def}}{=} \{1\} \cup \{i > 1 : \sigma(i-1) \geq \sigma(i) + k \text{ or } \sigma(i-1) < k\} \quad (3.6)$$

Why is this set important? It is exactly the set of positions such that des_k does not increase when we insert at that position. More precisely,

- If $i \in A_k(\sigma)$, then $\text{des}_k(\text{insert}_i(\sigma)) = \text{des}_k(\sigma)$.
- If $i \notin A_k(\sigma)$, then $\text{des}_k(\text{insert}_i(\sigma)) = \text{des}_k(\sigma) + 1$.

To see why, note that when we insert into σ at position i , we may gain one k -descent if $\sigma(i-1) \geq k$. However, if this is the case, we may also lose one descent (for a net gain of 0) if $\sigma(i-1) \geq \sigma(i) + k$ was already a k -descent.

Example 3.4. Consider the permutation $\sigma = 5 \ 2 \ 1 \ 3 \ 4$ and $k = 2$. First increment each number:

$$6 \quad 3 \quad 2 \quad 4 \quad 5$$

Case 1. $\sigma(i-1) < k$ and $i \in A_k(\sigma)$. For example, take $i = 4$.

$$6 \quad 3 \quad 2 \quad 1 \quad 4 \quad 5$$

Here, des_k does not increase because $2 \ 1$ is not a 2-descent. Thus, $4 \in A_k(\sigma)$ since $\sigma(3) = 1 < 2$.

Case 2. $\sigma(i-1) \geq \sigma(i) + k$ and $i \in A_k(\sigma)$. For example, take $i = 2$.

$$6 \quad 1 \quad 3 \quad 2 \quad 4 \quad 5$$

Here, des_k does not increase because, although $6 \ 1$ is indeed a 2-descent, there was already a 2-descent, $6 \ 3$ that gets broken by the inserted 1.

Case 3. $i \notin A_k$. For example, take $i = 3$.

$$6 \quad 3 \quad 1 \quad 2 \quad 4 \quad 5$$

Here, des_k does increase because $3 \ 1$ is a 2-descent. We did not lose any 2-descents, since $3 \ 2$ was not a 2-descent.

Now we will see how this set $A_k(\sigma)$ relates to $\widetilde{\text{maj}}_k(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_k(\sigma)$.

Lemma 3.5. For $\sigma \in \mathfrak{S}_{n-1}$, we have

$$\widetilde{\text{maj}}_k(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_k(\sigma) = |A_k(\sigma) \cup \{i+1, \dots, n\}| + \begin{cases} 0 & \text{if } i \in A_k(\sigma) \\ i-1 & \text{if } i \notin A_k(\sigma) \end{cases}$$

Proof. Recall from definition (1.9) that

$$\widetilde{\text{maj}}_k(\sigma) = \text{maj}_k(\sigma) + \#\{(i, j) : i < j \text{ and } \sigma(i) < \sigma(j) < \sigma(i) + k\}. \quad (3.7)$$

Let

$$S_k(\sigma) \stackrel{\text{def}}{=} \{(i, j) : i < j \text{ and } \sigma(i) < \sigma(j) < \sigma(i) + k\} \quad (3.8)$$

be the second component of that sum. We consider $\text{maj}_k(\text{insert}_i(\sigma)) - \text{maj}_k(\sigma)$ and $|S_k(\text{insert}_i(\sigma))| - |S_k(\sigma)|$ separately.

We can see that

$$|S_k(\text{insert}_i(\sigma))| - |S_k(\sigma)| = \#\{j \geq i : \sigma(j) < k\} \quad (3.9)$$

since each such j in the set means we have a pair $(i, j+1)$ for which $i < j+1$ and $\text{insert}_i(\sigma)(i) = 1 < \text{insert}_i(\sigma)(j) < k+1$.

We also have that

$$\text{maj}(\text{insert}_i(\sigma)) - \text{maj}(\sigma) = \#\{j \geq i : \sigma(j) \geq \sigma(j+1) + k\} + \begin{cases} 0 & \text{if } i \in A_k(\sigma) \\ i-1 & \text{if } i \notin A_k(\sigma) \end{cases} \quad (3.10)$$

because $\#\{j \geq i : \sigma(j) \geq \sigma(j+1) + k\}$ is the number of k -descents that get pushed one position to the right, and a k -descent is added at position $i-1$ if $i \notin A_k(\sigma)$. Adding (3.9) and (3.10) gives the result. \square

Given a set $A_k(\sigma)$ for a permutation $\sigma \in \mathfrak{S}_{n-1}$, we can construct a sequence $\mathcal{A}_k(\sigma) = a_0, \dots, a_{n-1}$

- $a_0 > \dots > a_{|A_k(\sigma)|-1}$ are the elements of $A_k(\sigma)$
- $a_{|A_k(\sigma)|} < \dots < a_{n-1}$ are the elements of $\{1, \dots, n\} \setminus A_k(\sigma)$, denotes $\overline{A}_k(\sigma)$.

Lemma 3.6. *For $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq j \leq n-1$, we have*

$$\widetilde{\text{maj}}_k(\text{insert}_{a_j}(\sigma)) - \widetilde{\text{maj}}_k(\sigma) = j$$

Proof. By Lemma (3.5) it suffices to show that $i = a_j$, then

$$j = |A_k(\sigma) \cup \{i+1, \dots, n\}| + \begin{cases} 0 & \text{if } i \in A_k(\sigma) \\ i-1 & \text{if } i \notin A_k(\sigma) \end{cases} \quad (3.11)$$

To see why this is true, we first consider the case that $i \in A_k(\sigma)$. In this case, $i = a_j$ is the $(j+1)^{\text{th}}$ largest element of $A_k(\sigma)$, so we get

$$j = |A_k(\sigma) \cup \{i+1, \dots, n\}|. \quad (3.12)$$

In the other case, suppose $i \notin A_k(\sigma)$. Since $a_{|A_k(\sigma)|}, \dots, a_{n-1}$ are the elements of $\overline{A}_k(\sigma)$, in ascending order, we have $j \geq |A_k(\sigma)|$ and $i = a_j$ is the $(j - |A_k(\sigma)| + 1)^{\text{th}}$ smallest element of $\overline{A}_k(\sigma)$. Hence,

$$j - |A_k(\sigma)| = |\overline{A}_k(\sigma) \cup \{1, \dots, i-1\}| \quad (3.13)$$

$$j - |A_k(\sigma)| = i - 1 - |A_k(\sigma) \cap \{1, \dots, i-1\}| \quad (3.14)$$

$$j = i - 1 + |A_k(\sigma)| - |A_k(\sigma) \cap \{1, \dots, i-1\}| \quad (3.15)$$

$$j = i - 1 + |A_k(\sigma) \cap \{i+1, \dots, n\}| \quad (3.16)$$

□

Lemma 3.6 shows that, as we claimed earlier, that $\widetilde{\text{maj}}_k(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_k(\sigma)$ ranges over $\{0, \dots, n-1\}$ as i ranges over $\{1, \dots, n\}$. Thus, we have shown that $\widetilde{\text{maj}}_k$ -Code satisfies clause (i) of Theorem 3.1.

3.4 Analysis of des_k and $\widetilde{\text{des}}_{k+1}$

We now turn our attention towards proving that $\widetilde{\text{maj}}_k$ -Code satisfies (ii) and (iii) of Theorem 3.1.

Theorem 3.1 (ii). We want to show that $\text{des}_k(\sigma) = \text{st}_k(\widetilde{\text{maj}}\text{-Code}(\sigma))$, for $\sigma \in \mathfrak{S}_n$. We will prove this by induction on n . The claim is obvious for $n \leq k$, since in that case we will always have $\text{des}_k(\sigma) = 0$ and $\text{st}_k(\sigma) = 0$. Now suppose that $n > k$, and suppose that we know that it is true for $n-1$. Let $\sigma \in \mathfrak{S}_{n-1}$. We have already established that $\text{des}(\text{insert}_i(\sigma)) = \text{des}(\sigma)$ if $i \in A_k(\sigma)$ and $\widetilde{\text{des}}(\text{insert}_i(\sigma)) = \widetilde{\text{des}}(\sigma) + 1$ otherwise. But we also know from Lemma 3.6 that $i \in A_k(\sigma)$ if and only if $\widetilde{\text{maj}}_k(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_k(\sigma) < |A_k(\sigma)| = \text{des}(\sigma) + k$. This is the same recurrence that st_k follows, and thus by induction, (ii) holds. □

Theorem 3.1 (iii). We want to show that $\widetilde{\text{des}}_{k+1}(\sigma) = \text{st}_k(\widetilde{\text{maj}}\text{-Code}(\sigma))$, for $\sigma \in \mathfrak{S}_n$. Once again, we will use induction, noting that $\widetilde{\text{des}}_{k+1}(\sigma) = 0$ for $n \leq k$. Assume $\sigma \in \mathfrak{S}_{n-1}$, with $n > k$.

We have the set A_k which allows us to study des_k . We will define a related set \widetilde{A}_k which will allow us to study $\widetilde{\text{des}}_{k+1}$. For any $\sigma \in \mathfrak{S}_{n-1}$, let

$$\widetilde{A}_{k+1}(\sigma) \stackrel{\text{def}}{=} \begin{cases} A_{k+1}(\sigma) & \text{if } \sigma(1) > n-k \\ A_{k+1}(\sigma) \setminus \{1\} & \text{otherwise} \end{cases} \quad (3.17)$$

First, we claim that if $\widetilde{\text{des}}_{k+1}(\text{insert}_i(\sigma)) - \widetilde{\text{des}}_{k+1}(\sigma)$ is 0 is $i \in \tilde{A}_{k+1}(\sigma)$, and 1 otherwise. For $i > 1$, this is true for the same reason that $A_{k+1}(\sigma)$ applies to des_{k+1} . Suppose $i = 1$. Then we will get $\text{insert}_i(\sigma)(1) = 1$. Since $n > k$, we will get

$$\widetilde{\text{des}}_{k+1}(\text{insert}_i(\sigma)) = \text{des}_{k+1}(\text{insert}_i(\sigma)) + 1 \quad (3.18)$$

$$= \text{des } k + 1(\sigma) + 1 \quad (3.19)$$

Furthermore,

$$\widetilde{\text{des}}_{k+1}(\sigma) = \text{des } k + 1(\sigma) + 1$$

if $\sigma(1) \leq n - k$ and

$$\widetilde{\text{des}}_{k+1} = \text{des}_{k+1}(\sigma)$$

otherwise. Thus, if $i = 1$, $\widetilde{\text{des}}_{k+1}$ increases by 1 if and only if $1 \in \tilde{A}_k(\sigma)$.

Our next step is to show that $i \in \tilde{A}_k(\sigma)$ if and only if

$$\widetilde{\text{maj}}_{k+1}(\text{insert}_i(\sigma)) - \widetilde{\text{maj}}_{k+1}(\sigma) < \widetilde{\text{des}}_{k+1}(\sigma) + k. \quad (3.20)$$

This will complete the proof as it will show $\widetilde{\text{des}}_{k+1}$ follows from the same recurrence as st_k . Thus, we just want to show that $j < \widetilde{\text{des}}_{k+1}(\sigma) + k$ if and only if $i \in \tilde{A}_{k+1}(\sigma)$. However, it is easy to see that this is the case, as the elements of $\tilde{A}_k(\sigma)$ are the first $|\tilde{A}_k(\sigma)| = \widetilde{\text{des}}_{k+1}(\sigma) + k$ elements of the sequence (a_j) . This is because the first $|A_k(\sigma)|$ elements of the sequence are the elements of $A_k(\sigma)$. Thus the result is immediately true if $A_k(\sigma) = \tilde{A}_k(\sigma)$. It is still true even if the 1 is missing from $\tilde{A}_k(\sigma)$, since 1 is the smallest element of $A_k(\sigma)$, and thus appears as the last in the sequence among the elements of $A_k(\sigma)$. \square

4 A quasi-symmetric function proof

In this section, we present another proof of Proposition 1.1 using quasi-symmetric functions suggested by Michelle Wachs. First, recall the definitions in Section 1 for des , maj , des_2 , maj_2 , $\widetilde{\text{des}}_2$. We also defined asc and ASC in Section 1, now similar as for des , we define $\text{amaj}(\sigma) = \sum_{i \in \text{ASC}(\sigma)} i$, $\text{ASC}_2(\sigma) = \{i \mid \sigma(i) < \sigma(i+1) - 1\}$, $\text{asc}_2(\sigma) = |\text{ASC}_2(\sigma)|$, $\text{amaj}_2(\sigma) = \sum_{i \in \text{ASC}_2(\sigma)} i$, and

$$\widetilde{\text{asc}}_2(\sigma) = \begin{cases} \text{asc}_2(\sigma) & \text{if } \sigma(1) = 1 \\ \text{asc}_2(\sigma) + 1 & \text{if } \sigma(1) \neq 1 \end{cases}$$

Similar as $\widetilde{\text{des}}_2$, we can see that $\widetilde{\text{asc}}_2$ is also equal distributed as cover over all permutations of n letters. The main result of this section is the following. Notice that if we let $q = 1$ in the following theorem, we get Proposition 1.1.

Theorem 4.1. *For any $n \geq 1$, we have*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{amaj}_2(\sigma)} p^{\widetilde{\text{asc}}_2(\sigma)} t^{\text{des}(\sigma^{-1})} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)}$$

To prove Theorem 4.1, we first state a few results we need about quasi-symmetric functions. Here we will skip the detailed definitions, which can be found in the corresponding references of the following results.

Theorem 4.2 ([12], (4.8)). *For any $n \geq 1$, we have*

$$\sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{DEX}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{DES}_2(\sigma)} t^{\text{des}(\sigma^{-1})},$$

where $\text{DEX}(\sigma)$ is some statistic related with DES . We will only need the following description of DEX .

Lemma 4.3 ([13], Lemma 2.2).

$$\sum_{i \in \text{DEX}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma),$$

$$|\text{DEX}(\sigma)| = \begin{cases} \text{des}(\sigma) & \text{if } \sigma(1) = 1 \\ \text{des}(\sigma) - 1 & \text{if } \sigma(1) \neq 1 \end{cases}$$

Proof of Theorem 4.1. By palindromicity, we have

$$\sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{DES}_2(\sigma)} t^{\text{des}(\sigma^{-1})} = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{ASC}_2(\sigma)} t^{\text{des}(\sigma^{-1})}$$

Thus by Theorem 4.2 we have

$$\sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{DEX}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{ASC}_2(\sigma)} t^{\text{des}(\sigma^{-1})}.$$

Now apply specializations to the above. By Lemma 5.2 of [6] (see Lemma 2.1 of [13]), we get

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\Sigma \text{DEX}(\sigma)} p^{|\text{DEX}(\sigma)|} t^{\text{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\Sigma \text{ASC}_2(\sigma)} p^{\text{asc}_2(\sigma)} t^{\text{des}(\sigma^{-1})}.$$

Apply Lemma 4.3.

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)} + \sum_{\sigma \in \mathfrak{S}_n : \sigma(1) \neq 1} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} p^{\text{des}(\sigma) - 1} t^{\text{exc}(\sigma)} = \\ & \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1} q^{\text{amaj}_2(\sigma)} p^{\widetilde{\text{asc}_2}(\sigma)} t^{\text{des}(\sigma^{-1})} + \sum_{\sigma \in \mathfrak{S}_n : \sigma(1) \neq 1} q^{\text{amaj}_2(\sigma)} p^{\widetilde{\text{asc}_2}(\sigma) - 1} t^{\text{des}(\sigma^{-1})} \end{aligned}$$

Hence we just need to show that

$$\sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1} q^{\text{amaj}_2(\sigma)} p^{\widetilde{\text{asc}_2}(\sigma)} t^{\text{des}(\sigma^{-1})}.$$

If we assume by induction that this is true for $n-1$, then we have

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} (pq)^{\text{des}(\sigma)} t^{\text{exc}(\sigma)} \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\text{amaj}_2(\sigma)} (pq)^{\widetilde{\text{asc}_2}(\sigma)} t^{\text{des}(\sigma^{-1})} \end{aligned}$$

But then

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\text{amaj}_2(\sigma)} (pq)^{\widetilde{\text{asc}_2}(\sigma)} t^{\text{des}(\sigma^{-1})} \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1} : \sigma(1)=1} q^{\text{amaj}_2(\sigma)} (pq)^{\text{asc}_2(\sigma)} t^{\text{des}(\sigma^{-1})} + \sum_{\sigma \in \mathfrak{S}_{n-1} : \sigma(1) \neq 1} q^{\text{amaj}_2(\sigma)} (pq)^{\text{asc}_2(\sigma) + 1} t^{\text{des}(\sigma^{-1})} \\ &= \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1, \sigma(2)=2} q^{\text{amaj}_2(\sigma)} p^{\text{asc}_2(\sigma)} t^{\text{des}(\sigma^{-1})} + \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1, \sigma(2) \neq 2} q^{\text{amaj}_2(\sigma)} p^{\text{asc}_2(\sigma)} t^{\text{des}(\sigma^{-1})} \\ &= \sum_{\sigma \in \mathfrak{S}_n : \sigma(1)=1} q^{\text{amaj}_2(\sigma)} p^{\widetilde{\text{asc}_2}(\sigma)} t^{\text{des}(\sigma^{-1})}. \end{aligned}$$

So we are done. □

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